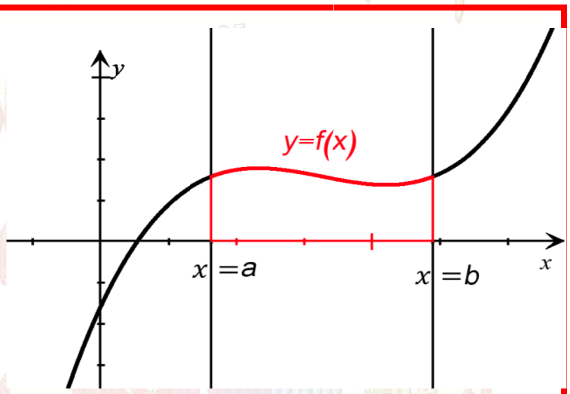
HOSSAM GHANEM

(31) 9.1 Arc Length

The length of the curve $y = f(x) \quad a \le x \le b$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



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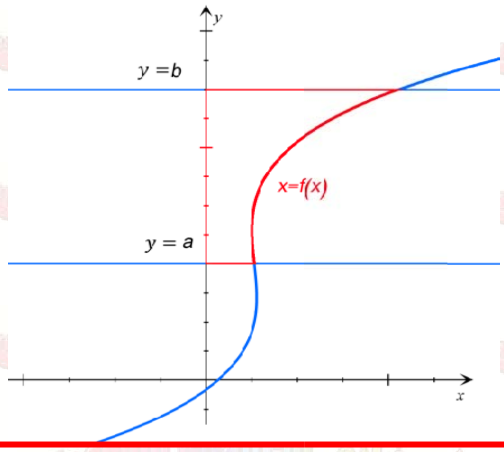

Example 1

Let the curve C : $y = \frac{1}{4}x^3 + \frac{1}{3x}$, $x \in [\,1\,,2\,]$ Find its exact length

Solution

$$y = \frac{1}{4}\,x^3 + \frac{1}{3x}$$

$$\frac{dy}{dx} = \frac{3}{4}\,x^2 - \frac{1}{3x^2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{3}{4}\,x^2\right)^2 + \left(\frac{1}{3x^2}\right)^2 - \frac{1}{2}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{3}{4}\,x^2\right)^2 + \left(\frac{1}{3x^2}\right)^2 + \frac{1}{2} = \left(\frac{3}{4}\,x^2 + \frac{1}{3x^2}\right)^2$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{3}{4}\,x^2 + \frac{1}{3x^2}$$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; dx$$

$$= \int_1^2 \left(\frac{3}{4}\,x^2 + \frac{1}{3x^2}\right) dx = \int_1^2 \left(\frac{3}{4}\,x^2 + \frac{1}{3}\,x^{-2}\right) dx = \left[\frac{1}{4}\,x^3 + \frac{-1}{3}\,x^{-1}\right]_{-1}^{2} = \left[\frac{x^3}{4} - \frac{1}{3x}\right]_{-1}^{2}$$

$$= \left(\frac{8}{4} - \frac{1}{6}\right) - \left(\frac{-1}{4} - \frac{1}{-3}\right) = 2 - \frac{1}{6} + \frac{1}{4} - \frac{1}{3} = \frac{24 - 2 + 3 - 4}{12} = \frac{21}{12}$$

Example 2

Let the curve C : $y = \ln(\cos x)$, $x \in \left[\,0\,,\frac{\pi}{3}\right]$. Find its exact length

Solution

$$y = \ln(\cos x)$$

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$$

$$\left(\frac{dy}{dx}\right)^2 = \tan^2 x$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sec x$$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; dx$$

$$L = \int_0^{\frac{\pi}{3}} \sec x\;\; dx = \Big[\ln|\sec x + \tan x|\Big]_0^{\frac{\pi}{3}} = \ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \ln|\sec 0 + \tan 0|$$

$$= \ln|2 + \sqrt{3}| - \ln|1 - 0| = \ln|2 + \sqrt{3}|$$

Example 3

Let the curve C : $y^2 = 4x$, $y \in [\,0\,, 2\,]$ Find its exact length

Solution

$y^2 = 4x$

$x = \frac{1}{4} y^2$

$$\frac{dy}{dx} = \frac{1}{2} y$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4} y^2$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{4} y^2$$

$$L = \int_c^d \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; dy$$

$$L = \int_0^2 \sqrt{1 + \frac{1}{4} y^2}\; dy$$

Let $y = 2\tan\theta$ ⇨ $dy = 2\sec^2\theta \; d\theta$

at $y = 0$ $2\tan\theta = 0$ $\theta = 0$

at $y = 2$ $2\tan\theta = 2$ $\tan\theta = 1$ $\theta = \frac{\pi}{4}$

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1 + \frac{1}{4} \cdot 4\tan^2\theta} \;\cdot\; 2\sec^2\theta \; d\theta$$

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2\theta} \;\cdot\; 2\sec^2\theta \; d\theta = 2\int_0^{\frac{\pi}{4}} \sec^3\theta \; d\theta$$

$$= 2 \cdot \frac{1}{2} \Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_0^{\frac{\pi}{4}}$$

$$L = \Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_0^{\frac{\pi}{4}}$$

$$= \left[\sqrt{2}(1) + \ln\left(\sqrt{2} + 1\right)\right] - \left[(1)(0) + \ln(1 + 0)\right]$$

$$= \sqrt{2} + \ln\left(\sqrt{2} + 1\right)$$

$$I_1 = \int \sec^3\theta \; d\theta$$

$u = \sec\theta$ $dv = \sec^2\theta \; d\theta$

$du = \sec\theta\tan\theta \; d\theta$ $v = \tan\theta$

$$I_1 = uv - \int v\,du$$

$$I_1 = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta \; d\theta$$

$$= \sec\theta\tan\theta - \int \sec\theta \; (\sec^2\theta - 1)\, d\theta$$

$$= \sec\theta\tan\theta - \int (\sec^3\theta - \sec\theta)\, d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta \; d\theta + \int \sec\theta \; d\theta$$

$$= \sec\theta\tan\theta + \int \sec\theta \; d\theta - I_1$$

$$2\, I_1 = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|$$

$$I_1 = \frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]$$

Example 4

Let the curve C : $y = \ln\left(\dfrac{e^x+1}{e^x-1}\right)$, $x \in [\ln 2\ , \ln 4\ ]$ Find its exact length

Solution

$$y = \ln(e^x+1) - \ln(e^x-1)$$

$$\frac{dy}{dx} = \frac{e^x}{e^x+1} - \frac{e^x}{e^x-1} = \frac{e^x(e^x-1) - e^x(e^x+1)}{(e^x+1)(e^x-1)} = \frac{e^x[e^x-1-e^x-1]}{e^{2x}-1}$$

$$= \frac{-2\,e^x}{e^{2x}-1} = \frac{-2}{e^x-e^{-x}} = -\operatorname{csch} x$$

$$\left(\frac{dy}{dx}\right)^2 = \operatorname{csch}^2 x$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\operatorname{csch}^2 x = \coth^2 x$$

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$L = \int_{\ln 2}^{\ln 4} \coth x\ \ dx = \int_{\ln 2}^{\ln 4} \frac{\cosh x}{\sinh x}\ dx$$

$$= \Big[\ln|\sinh x|\Big]_{\ln 2}^{\ln 4} = \ln(\sinh(\ln 4)) - \ln(\sinh(\ln 2))$$

$$= \ln\frac{1}{2}\left(e^{\ln 4} - e^{-\ln 4}\right) - \ln\frac{1}{2}\left(e^{\ln 2} - e^{-\ln 2}\right)$$

$$= \ln\frac{1}{2}\left(4-\frac{1}{4}\right) - \ln\frac{1}{2}\left(2-\frac{1}{2}\right) = \ln\frac{15}{8} - \ln\frac{3}{4} = \ln\left(\frac{15}{8}\cdot\frac{4}{3}\right) = \ln\frac{5}{2}$$

Homework

1	Let the curve C : $y=\frac{1}{2}x^2-\frac{\ln x}{4}$, $x\in[2,4]$. Find its exact length
2	Let the curve C : $y=\frac{1}{8}x^4+\frac{1}{4x^2}$, $x\in[1,2]$. Find its exact length
3	Let the curve C : $y=\frac{1}{6}x^5+\frac{1}{10x^3}$, $x\in[1,2]$. Find its exact length
4	Let the curve C : $y=\ln(\sec x)$, $x\in\left[0,\frac{\pi}{4}\right]$. Find its exact length
5	Let the curve C : $y=\cosh x$, $x\in[0,1]$. Find its exact length
6	Find the length of the curve $y=\ln\sqrt{\tanh x}$, $1\le x\le 2$ (4 points) 37 August 7, 2010